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J. Math. Anal. Appl. 293 (2004) 645–662

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Exponential bounds for discrete-time singularly perturbed Markov chains ☆

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Received 19 March 2003

Submitted by S. Sethi

Abstract

This paper develops exponential type upper bounds for scaled occupation measures of singularly perturbed Markov chains in discrete time. By considering two-time scale in the Markov chains, asymptotic analysis is carried out. The cases of the fast changing transition probability matrix is irreducible and that are divisible into l ergodic classes are examined first; the upper bounds of a sequence of scaled occupation measures are derived. Then extensions to Markov chains involving transient states and/or nonhomogeneous transition probabilities are dealt with. The results enable us to further our understanding of the underlying Markov chains and related dynamic systems, which is essential for solving many control and optimization problems.

Keywords: Markov chain; Singular perturbation; Exponential bound

1. Introduction

This work is concerned with discrete-time singularly perturbed Markov chains. Our effort is devoted to obtaining exponential upper bounds for a scaled sequence of occupation measures. The study stems from a wide range of applications in control and optimization of large-scale hybrid systems. By hybrid systems, we mean such systems in which the usual dynamics are intertwined with jump Markov chains. Some of the recent development on

☆ The research was supported in part by the National Science Foundation.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.01.025

the subject and its various applications, we refer the reader to [1–5,12,14,18]. For related numerical results and simulations of singularly perturbed Markov chains and the associated control problems, see [9,19]. For references on singular perturbation, see [10,11,13].

In many problems arising from wireless communication, signal processing, and financial engineering, such as those in queuing systems, ATM (asynchronous transmission mode), CDMA/DS (code-division multiple-access/direct sequence), etc., the discrete events usually possess large state spaces due to the consideration of various factors imbedded in the problem. The hybrid formulation makes a more realistic model, but it also poses new challenges. We have to device feasible approaches to treat the large-scale systems.

As observed in [15] (see also [17,20]), in a large-scale system, different parts may exhibit rates of changes in high contrast. Some of them vary rapidly and others evolve slowly. One introduces a two-time scale into the formulation naturally. The different rates of changes allow us to take advantages of the inherent hierarchical structure, and obtain a limit system that is averaged out with respect to certain invariant measures. For example, if we want to find the optimal control of a large-scale hybrid system, the amount of computation needed may be infeasible. To reduce the computation complexity, we can use the optimal or near-optimal control of the limit system to construct controls of the original system leading to near optimality. To be able to understand the basic properties of the singularly perturbed systems is the key to many control and optimization problems.

In our recent work, we have examined asymptotic properties of singularly perturbed Markov chains in discrete time. Specifically, we have obtained the asymptotic expansions of the probability vectors and transition matrices [21], studied aggregations of the underlying processes, and obtained switching diffusion limit of a sequence of scaled occupation measures [23] (see also [22]). These results are helpful for further investigation in many control and optimization problems. However, they do not provide estimate of rare event probabilities. In this paper, we focus on the exponential type bounds for the singularly perturbed systems. Such a study is important to further our understanding of system stability, and to facilitate the calculation of certain probabilities of rare events.

Example 1.1 (*Risk-sensitive production control*). Consider a manufacturing system that produces a single product type using m identical machines. Let $\alpha_k^\varepsilon \in \{1, 2, \dots, m\}$ denote the system capacity process. Assume α_k^ε is a Markov chain with transition matrix P^ε. Here ε is a small parameter. Let u_k denote the rate of production, x_k denote the surplus, and z_k denote demand rate. They satisfy the following difference equation:

$$x_{k+1} = x_k + \varepsilon(u_k - z_k).$$

The objective of the problem is to choose u_k over time subject to production constraint

$$0 \leqslant u_k \leqslant \alpha_k^\varepsilon,$$

to minimize

$$J^\varepsilon = \sqrt{\varepsilon} \log E\left[\exp\left(\frac{\theta}{\sqrt{\varepsilon}} \sum_{k=0}^{\infty} \rho^k \big(h(x_k) + c(u_k)\big)\right)\right],$$

where $\theta > 0$ is a given number, $0 < \rho < 1$ is a discount factor, $h(\cdot)$ and $c(\cdot)$ are inventory and production costs, respectively.

The above problem formulation using the exponential cost function emphasizes the stability and penalizes heavily on sample paths of x_k and u_k that have significant contribution to the cost function. It is well known that exact optimal solution for the problem is difficult to obtain. In order to implement a control policy in practice, one has to resort to approximate solutions. In this connection, exponential error bounds to be derived in this paper provide guidelines for getting error estimates needed in near optimality. See [24] for results in continuous-time setting.

Singular perturbation of discrete-time Markov chains is closely related to convergence of stochastic matrices. In [7], stationary distribution of a perturbed stochastic matrix is studied under matrix commutative conditions. In [6], a truncated ergodic average of a singularly perturbed stochastic matrix is considered. Sufficient conditions are provided that guarantee the convergence of the average. For a recent survey on singularly perturbed Markov decision processes and their perturbed stationary distribution matrices, see [4].

The rest of the paper is arranged as follows. Section 2 begins with the formulation of the problem. Section 3 concentrates on the case that the fast changing part of the transition probability matrix is irreducible, which is a necessary step for studying further properties of Markov chains with more complex structure. Section 4 considers the case that the fast changing transition matrix is no longer irreducible, but rather decomposable into l subtransition matrices corresponding to the decomposition of the state space into l irreducible classes. Section 5 gives examples on applications of these error bounds. Extensions that incorporate time dependence and that include transient states are given in Section 6. Finally, the paper is closed with some further remarks.

2. Formulation

Suppose T is a positive real number, $\varepsilon > 0$ is a small parameter, and α_k^ε, for $0 \leqslant k \leqslant \lfloor T/\varepsilon \rfloor$, is a discrete-time Markov chain with finite state space $\mathcal{M} = \{1, \dots, m\}$, where $\lfloor z \rfloor$ denotes the integer part of a real number z. For notational simplicity, in what follows, we often simply write T/ε in lieu of $\lfloor T/\varepsilon \rfloor$. Throughout the paper, for a discrete-time singularly perturbed process, we work with the time horizon $0 \leqslant k \leqslant T/\varepsilon$, whereas for a continuous-time process, we work with the time horizon $[0, T]$.

Consider a discrete-time Markov chain α_k^ε with stationary transition probability matrix P^ε given by

$$P^\varepsilon = P + \varepsilon Q, \tag{2.1}$$

where $P = (p_{ij})$ itself is a transition probability matrix, i.e., $p_{ij} \geqslant 0$ for each i, j and $\sum_j p_{ij} = 1$, and $Q = (q_{ij})$ is a generator, i.e., for each $i \neq j$, $q_{ij} \geqslant 0$ and for each i, $\sum_j q_{ij} = 0$ or $Q\mathbb{1}_m = 0$. Here and hereafter, $\mathbb{1}_\iota \in \mathbb{R}^{\iota \times 1}$ denotes a column vector with all components being 1.

We are interested in the cases (see, for example, [8, p. 94]) a transition probability matrix of a finite-state Markov chain can be put into the form

$$P = \operatorname{diag}(P_1, \dots, P_l), \tag{2.2}$$

where for each $i \leqslant l$, P_i is a transition matrix within the ith recurrent class. Here and hereafter, $\operatorname{diag}(A_1, \dots, A_j)$ denotes a block diagonal matrix with matrix entries $A_1, \dots, A_j$ having appropriate dimensions. A Markov chain with transition matrix given by (2.2) consists of l recurrent classes, whereas a Markov chain with transition matrix (6.3) has l recurrent classes and a number of transient states. Let $\mathcal{M}_i = \{s_{i1}, \dots, s_{im_i}\}$ denote the states corresponding to the ith block P_i. Then the state space of α_k^ε is $\mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l$.

To study the exponential bounds of a sequence of occupation measures for the Markov chain α_k^ε with transition matrix P given by (2.2), we need to first derive the corresponding bounds for P being an irreducible matrix. Then such a result will help us to further derive the bounds for the more complex cases.

2.1. Preliminary results

Suppose that α_k^ε has transition probabilities (2.1) with P having a block diagonal form (2.2). For some $T > 0$ and for each $k = 0, 1, \dots, T/\varepsilon$, the probability vector

$$x_k^\varepsilon = \left(x_{k,1}^\varepsilon, \dots, x_{k,m}^\varepsilon\right) = \left(P\left(\alpha_k^\varepsilon = 1\right), \dots, P\left(\alpha_k^\varepsilon = m\right)\right) \in \mathbb{R}^{1\times m}$$

satisfies the vector-valued difference equation

$$x_{k+1}^\varepsilon = x_k^\varepsilon P^\varepsilon, \qquad x_0^\varepsilon = x_0 \tag{2.3}$$

such that each component $x_{0,i} \geqslant 0$ and $x_0 \mathbb{1}_m = \sum_{i=1}^m x_{0,i} = 1$. Note that x_0 is independent of ε and is the initial probability distribution. To proceed, we make the following assumption.

(A) P^ε, P, and P_i for $i \leqslant l$ are transition probability matrices such that for each $i \leqslant l$, P_i is aperiodic and irreducible.

We obtain asymptotic expansions of the probability vector and the corresponding transition probabilities using Fredholm alternative via verification of orthogonality conditions (see [21]). In addition, many applications require the understanding of the amount of time the Markov chain spends in a given state. This leads to the study of occupation measures. Define $\bar{\alpha}_r^\varepsilon = i$ if $\alpha_r^\varepsilon \in \mathcal{M}_i$. For $k = 0, \dots, T/\varepsilon$, $i = 1, \dots, l$, and $j = 1, \dots, m_i$, define a sequence of occupation measures

$$\begin{aligned} o_{k,ij}^\varepsilon &= \varepsilon \sum_{r=1}^{k} \left(I_{\{\alpha_r^\varepsilon = s_{ij}\}} - \nu_j^i I_{\{\bar{\alpha}_r^\varepsilon = i\}}\right), \\ O_k^\varepsilon &= \left(o_{k,11}^\varepsilon, \dots, o_{k,1m_1}^\varepsilon, \dots, o_{k,l1}^\varepsilon, \dots, o_{k,lm_l}^\varepsilon\right). \end{aligned} \tag{2.4}$$

The following lemma summaries the results of asymptotic expansions and mean square bounds on occupation measures.

Lemma 2.1. *Assume condition* (A). *Then the following assertions hold*:

(a) *For the probability distribution vector x_k^ε, we have*

$$x_k^\varepsilon = \theta(\varepsilon k) \operatorname{diag}\left(\nu^1, \dots, \nu^l\right) + O\left(\varepsilon + \lambda^k\right), \quad \text{for } 0 \leqslant k \leqslant T/\varepsilon, \tag{2.5}$$

for some λ with $0 < \lambda < 1$, where ν^i is the stationary distribution corresponding to the transition matrix P_i, and $\theta(t) = (\theta_1(t), \dots, \theta_l(t)) \in \mathbb{R}^{1\times l}$ satisfies

$$\frac{d\theta(t)}{dt} = \theta(t)\overline{Q}, \qquad \theta_i(0) = x_0^i \mathbb{1}_{m_i},$$

where

$$\overline{Q} = \mathrm{diag}\big(\nu^1, \dots, \nu^l\big) Q\tilde{\mathbb{1}}, \qquad \tilde{\mathbb{1}} = \mathrm{diag}(\mathbb{1}_{m_1}, \dots, \mathbb{1}_{m_l}). \tag{2.6}$$

(b) *For $k \leqslant T/\varepsilon$, the k-step transition probability matrix $(P^\varepsilon)^k$ satisfies*

$$\big(P^\varepsilon\big)^k = \Phi(\varepsilon k) + \varepsilon\hat{\Phi}(\varepsilon k) + \Psi(k) + \varepsilon\hat{\Psi}(k) + O\big(\varepsilon^2\big), \tag{2.7}$$

where

$$\begin{aligned}
&\Phi(t) = \tilde{\mathbb{1}}\Theta(t)\,\mathrm{diag}\big(\nu^1, \dots, \nu^l\big),\\
&\frac{d\Theta(t)}{dt} = \Theta(t)\overline{Q}, \qquad \Theta(0) = I,\\
&\big|\Psi(k)\big| \leqslant K\lambda^k \quad \text{and} \quad \big|\hat{\Psi}(k)\big| \leqslant K\lambda^k.
\end{aligned} \tag{2.8}$$

(c) *Then for $i = 1, \dots, l$, $j = 1, \dots, m_i$,*

$$\sup_{0\leqslant k\leqslant T/\varepsilon} E\big|o^\varepsilon_{k,ij}\big|^2 = O(\varepsilon).$$

Sketch of Proof. We provide the sketch of proof only because the details for parts (a) and (b) follows from [21, Theorems 3.5 and 4.3] and that for part (c) can be found in [23, Theorem 3.2].

To see part (a), using [21, Theorem 4.3], one can write x_k^ε as the zero-order term of the asymptotic expansion $\theta(\varepsilon k)\,\mathrm{diag}(\nu^1, \dots, \nu^l)$ plus an exponential decay term of order $O(\lambda^k)$ with uniform error bound $O(\varepsilon)$. This yields (2.5). Similarly, one can prove part (b), which is a matrix form of the approximation in part (a).

Part (c) can be derived using the following steps. Write $E|o^\varepsilon_{k,ij}|^2$ in terms of the corresponding conditional probabilities explicitly; apply asymptotic expansion result in part (b); simplify the resulting expression leading to the desired order estimate. □

3. Exponential bounds under irreducibility

This section is devoted to the exponential bounds of such Markov chains whose transition matrix is given by (2.1) with P being irreducible. Such an upper bound is needed for studying cases with more general transition matrix P. It is also interesting in its own right.

Throughout this section, we assume the singularly perturbed Markov chain has a transition matrix P^ε given by (2.1), where P is an irreducible matrix. The state space $\mathcal{M} = \{1, \dots, m\}$. Let $\{\beta_k\}$ be a bounded sequence of real numbers. As in the study of asymptotic normality, define

$$n_k^\varepsilon = \big(n^\varepsilon_{k,1}, \dots, n^\varepsilon_{k,m}\big), \qquad n^\varepsilon_{k,i} = \sqrt{\varepsilon}\sum_{j=1}^k \big(I_{\{\alpha_j^\varepsilon = i\}} - \nu_i\big)\beta_j. \tag{3.1}$$

Define also

$$\chi_k^\varepsilon = \big(I_{\{\alpha_k^\varepsilon=1\}}, \ldots, I_{\{\alpha_k^\varepsilon=m\}}\big) \in \mathbb{R}^{1\times m},$$
$$w_k^\varepsilon = \chi_k^\varepsilon - \chi_0^\varepsilon - \sum_{j=0}^{k-1} \chi_j^\varepsilon (P + \varepsilon Q - I). \tag{3.2}$$

A simple calculation shows that

$$\chi_{k+1}^\varepsilon = \chi_k^\varepsilon + \chi_k^\varepsilon (P + \varepsilon Q - I) + \Delta w_k^\varepsilon, \tag{3.3}$$

where $\Delta w_k^\varepsilon = w_{k+1}^\varepsilon - w_k^\varepsilon$.

In view of Proposition 2.1, $P^\varepsilon \to \overline{P} = \mathbb{1}_m \nu$, as $\varepsilon \to 0$ and $k \to \infty$, where $\mathbb{1}_m = (1, \ldots, 1)' \in \mathbb{R}^{m\times 1}$ and $\nu = (\nu_1, \ldots, \nu_m)$ is the stationary distribution of P. Moreover, for $0 \leqslant k \leqslant T/\varepsilon$ for some finite $T > 0$, the k-step transition matrix $(P^\varepsilon)^k$ satisfies $(P^\varepsilon)^k - \overline{P} = O(\varepsilon + \lambda^k)$ for some $0 < \lambda < 1$. Denote the least upper bound of

$$\frac{(P^\varepsilon)^k - \overline{P}}{\varepsilon + \lambda^k}$$

by K_T for $0 \leqslant k \leqslant T/\varepsilon$. For convenience, introduce the notation $O_1(\cdot)$ as a "normalized" order symbol in that $O_1(y)$ is a function of y such that $|O_1(y)|/|y| \leqslant 1$. We then have

$$\big(P^\varepsilon\big)^k - \overline{P} = (P + \varepsilon Q)^k - \overline{P} = K_T O_1\big(\varepsilon + \lambda^k\big), \quad \text{for } 0 \leqslant k \leqslant T/\varepsilon. \tag{3.4}$$

For a vector $v = (v_i)$, we use $|\cdot|$ to denote the max norm

$$|v| = \max_i v_i,$$

and given a matrix $A = (a_{ij})$, we use the norm

$$|A| = \max_{i,j} |a_{ij}|.$$

With this notation, it is easy to show that

$$\big|\Delta w_k^\varepsilon\big| \leqslant 1.$$

Similarly, for a vector-valued sequence $z_k \in \mathbb{R}^{1\times m}$ and a matrix-valued sequence $A_k \in \mathbb{R}^{m\times m}$, we use

$$|z|_T = \sup_{0\leqslant k\leqslant T/\varepsilon} |z_k|, \qquad |A|_T = \sup_{0\leqslant k\leqslant T/\varepsilon} |A_k|, \tag{3.5}$$

respectively. We next derive the following exponential-type upper bounds on $n_{k,i}^\varepsilon$ defined in (3.1), which turns out to be useful for many control and optimization problems and for stability analysis.

Theorem 3.1. *Assume P is irreducible. Then there exists an $\varepsilon_0 > 0$ such that for all $0 \leqslant \varepsilon \leqslant \varepsilon_0$, the following error bound holds*:

$$\sup_{1\leqslant \kappa_\varepsilon \leqslant T/\varepsilon} E \exp\left(\frac{c_T}{(T+1)^{3/2}} \big|n_{\kappa_\varepsilon}^\varepsilon\big|\right) \leqslant K, \tag{3.6}$$

where n_k^ε *is defined in* (3.1), K *is a constant independent of* ε *and* T, c_T *is a constant satisfying*

$$0 \leqslant c_T \leqslant \frac{1-\lambda}{K_T(|\beta|_T + 1)}, \tag{3.7}$$

and λ *is as given in Lemma* 2.1.

Proof. It is easily verified that

$$w_k \overline{P} = \chi_k^\varepsilon \overline{P} - \chi_0^\varepsilon \overline{P} - \sum_{j=0}^{k-1} \chi_j^\varepsilon (P + \varepsilon Q - I) \overline{P} = 0. \tag{3.8}$$

Thus w_k is orthogonal to $\overline{P}$. Denote by $\mathcal{F}_k$ the σ-algebra generated by $\{\alpha_j^\varepsilon\colon\ j \leqslant k\}$. We next show that Δw_k^ε is a martingale difference sequence with respect to $\mathcal{F}_k$. In fact, for each fixed $j \in \mathcal{M}$, by the Markov property,

$$\begin{aligned} E\big(I_{\{\alpha_{k+1}^\varepsilon = j\}} | \mathcal{F}_k\big) = E\big(I_{\{\alpha_{k+1}^\varepsilon = j\}} | \alpha_k^\varepsilon\big) &= \sum_{i=1}^m I_{\{\alpha_k^\varepsilon = i\}} E\big(I_{\{\alpha_{k+1}^\varepsilon = j\}} | \alpha_k^\varepsilon = i\big) \\ &= \sum_{i=1}^m I_{\{\alpha_k^\varepsilon = i\}} p_{ij}^\varepsilon, \end{aligned}$$

so

$$E\big(\chi_{k+1}^\varepsilon | \mathcal{F}_k\big) = \left(\sum_{i=1}^m I_{\{\alpha_k^\varepsilon = i\}} p_{i1}^\varepsilon, \ldots, \sum_{i=1}^m I_{\{\alpha_k^\varepsilon = i\}} p_{im}^\varepsilon \right) = \chi_k^\varepsilon (P + \varepsilon Q).$$

As a result,

$$E(\Delta w_k | \mathcal{F}_k) = E\big(\chi_{k+1}^\varepsilon | \mathcal{F}_k\big) - \chi_k^\varepsilon (P + \varepsilon Q) = 0. \tag{3.9}$$

That is, the martingale difference property is verified.

By virtue of (3.3),

$$\chi_{k+1}^\varepsilon = \chi_0^\varepsilon (P + \varepsilon Q)^{k+1} + \sum_{j=0}^k \Delta w_j^\varepsilon (P + \varepsilon Q)^{k-j}. \tag{3.10}$$

Noting that $\chi_0^\varepsilon \overline{P} = \nu$ and the orthogonality of w_k and $\overline{P}$ (see (3.8)), we arrive at

$$\begin{aligned} \chi_{k+1}^\varepsilon - \nu &= \chi_0^\varepsilon \big[(P + \varepsilon Q)^{k+1} - \overline{P}\,\big] + \sum_{j=0}^k \Delta w_j^\varepsilon \big\{\big[(P + \varepsilon Q)^{k-j} - \overline{P}\,\big] + \overline{P}\,\big\} \\ &= \chi_0^\varepsilon \rho_{k+1} + \sum_{j=0}^k \Delta w_j^\varepsilon \rho_{k-j}, \end{aligned} \tag{3.11}$$

where

$$\rho_i = \big[(P + \varepsilon Q)^i - \overline{P}\,\big]. \tag{3.12}$$

Let $1 \leqslant \kappa_\varepsilon \leqslant T/\varepsilon$. It follows from (3.11),

$$\sum_{k=1}^{\kappa_\varepsilon}[\chi_k^\varepsilon - \nu]\beta_k = \sum_{k=0}^{\kappa_\varepsilon-1}[\chi_{k+1}^\varepsilon - \nu]\beta_{k+1} = \sum_{k=0}^{\kappa_\varepsilon-1}\chi_0^\varepsilon\rho_{k+1}\beta_{k+1} + \sum_{k=0}^{\kappa_\varepsilon-1}\left(\sum_{j=0}^{k}\Delta w_j^\varepsilon\rho_{k-j}\right)\beta_{k+1}. \tag{3.13}$$

Consider the last term in (3.13), interchanging the order of summation, we obtain

$$\sum_{k=0}^{\kappa_\varepsilon-1}\left(\sum_{j=0}^{k}\Delta w_j^\varepsilon\rho_{k-j}\right)\beta_{k+1} = \sum_{j=0}^{\kappa_\varepsilon-1}\Delta w_j^\varepsilon\tilde{\rho}_j, \quad \text{where } \tilde{\rho}_j = \sum_{i=0}^{\kappa_\varepsilon-1-j}\rho_i\beta_{i+j+1}.$$

Equation (3.4) implies that

$$|\tilde{\rho}_j| \leqslant K_T\left(T + \frac{1}{1-\lambda}\right)|\beta|_T, \quad \text{for all } j \leqslant T/\varepsilon.$$

It follows from (3.13) that

$$\left|n_{\kappa_\varepsilon}^\varepsilon\right| = \sqrt{\varepsilon}\left|\sum_{k=0}^{\kappa_\varepsilon-1}\left(\chi_{k+1}^\varepsilon - \nu\right)\beta_{k+1}\right| \leqslant \sqrt{\varepsilon}\sum_{k=0}^{\kappa_\varepsilon-1}\left|\chi_0^\varepsilon\rho_{k+1}\beta_{k+1}\right| + \sqrt{\varepsilon}\left|\sum_{k=0}^{\kappa_\varepsilon-1}\Delta w_j^\varepsilon\tilde{\rho}_j\right|.$$

In view of (3.4), we have

$$\sum_{k=0}^{\kappa_\varepsilon-1}\left|\chi_0^\varepsilon\rho_{k+1}\beta_{k+1}\right| \leqslant |\beta|_T K_T\sum_{k=0}^{\kappa_\varepsilon-1}\left(\varepsilon + \lambda^{k+1}\right) \leqslant |\beta|_T K_T\left(T + \frac{\lambda}{1-\lambda}\right). \tag{3.14}$$

Thus,

$$\exp\left(\frac{c_T\sqrt{\varepsilon}}{(T+1)^{3/2}}\sum_{k=0}^{\kappa_\varepsilon-1}\left|\chi_0^\varepsilon\rho_{k+1}\beta_{k+1}\right|\right) \leqslant \exp\left(\frac{c_T\sqrt{\varepsilon}}{(T+1)^{3/2}}|\beta|_T K_T\left(T + \frac{\lambda}{1-\lambda}\right)\right) \leqslant e,$$

for ε small enough. Let $p_k = (p_{k,1}, \ldots, p_{k,m})$ with

$$p_{k,i} = \sum_{j=0}^{k-1}\left(\frac{c_T}{T+1}\right)\Delta w_{j,i}\tilde{\rho}_j, \quad i \in \mathcal{M},$$

where $\Delta w_{j,i}$ denotes the ith component of Δw_j^ε. To complete the proof, it suffices to show

$$E\exp\left(\frac{\sqrt{\varepsilon}}{\sqrt{T+1}}|p_{\kappa_\varepsilon}|\right) \leqslant K. \tag{3.15}$$

Note that

$$Ee^\xi \leqslant e + (e-1)\sum_{j=1}^{\infty}e^j P(\xi \geqslant j), \tag{3.16}$$

for any nonnegative random variable ξ. Moreover, for each $a > 0$, we have, using the max norm,

$$P\big(|p_{\kappa_\varepsilon}| \geqslant a\big) = P\left(\bigcup_{i=1}^{m} |p_{\kappa_\varepsilon,i}| \geqslant a\right) \leqslant \sum_{i=1}^{m} P\big(|p_{\kappa_\varepsilon,i}| \geqslant a\big). \tag{3.17}$$

In view of these inequalities, (taking $\xi = (\sqrt{\varepsilon}/\sqrt{T+1})|p_{\kappa_\varepsilon}|$ in (3.16) and $a = j\sqrt{T+1}/\sqrt{\varepsilon}$ in (3.17)), it suffices to show

$$\sum_{j=1}^{\infty} e^j P\left(|p_{\kappa_\varepsilon,i}| \geqslant \frac{j\sqrt{T+1}}{\sqrt{\varepsilon}}\right) < \infty, \quad \text{for each } i \in \mathcal{M}. \tag{3.18}$$

In fact, we first note that, for $0 \leqslant k \leqslant T/\varepsilon$,

$$|\Delta p_{k,i}| = |p_{k+1,i} - p_{k,i}| = \left|\left(\frac{c_T}{T+1}\right)\Delta w_{k,i}\tilde{\rho}_k\right| \leqslant 1.$$

Given $i \in \mathcal{M}$ and $\zeta > 0$, define

$$q_k = 1 + \zeta \sum_{j=0}^{k-1} q_j \Delta p_{j,i}, \quad q_0 = 1.$$

Note that ζ is parameter to be chosen later to fit our needs. Then, $\{q_k\}$ is a martingale and $Eq_k = 1$. Moreover,

$$q_{k+1} - q_k = \zeta q_k \Delta p_{k,i}.$$

It follows that

$$q_k = \prod_{j=0}^{k-1} (1 + \zeta \Delta p_{j,i}).$$

Using the condition $|\Delta p_{j,i}| \leqslant 1$, it is easy to show that

$$1 + (\Delta p_{j,i})\zeta \geqslant e^{(\Delta p_{j,i})\zeta - \bar{\kappa}\zeta^2},$$

for some $\bar{\kappa} > 0$. This implies that

$$q_k \geqslant \prod_{j=0}^{k-1} e^{(\Delta p_{j,i})\zeta - \bar{\kappa}\zeta^2} = e^{p_{k,i}\zeta - \bar{\kappa}k\zeta^2}, \quad \text{for } k \leqslant T/\varepsilon.$$

Note that

$$P\left(|p_{\kappa_\varepsilon,i}| \geqslant \frac{j\sqrt{T+1}}{\sqrt{\varepsilon}}\right) = P\left(p_{\kappa_\varepsilon,i} \geqslant \frac{j\sqrt{T+1}}{\sqrt{\varepsilon}}\right) + P\left(-p_{\kappa_\varepsilon,i} \geqslant \frac{j\sqrt{T+1}}{\sqrt{\varepsilon}}\right).$$

We need only consider the first term because the second one can be treated similarly. We have

$$\begin{aligned} P\left(p_{\kappa_\varepsilon,i} \geqslant \frac{j\sqrt{T+1}}{\sqrt{\varepsilon}}\right) &\leqslant P\left(q_{\kappa_\varepsilon} \geqslant \exp\left(\frac{j\sqrt{T+1}}{\sqrt{\varepsilon}}\zeta - \bar{\kappa}\kappa_\varepsilon\zeta^2\right)\right) \\ &\leqslant \exp\left(-\frac{j\sqrt{T+1}}{\sqrt{\varepsilon}}\zeta + \bar{\kappa}\kappa_\varepsilon\zeta^2\right). \end{aligned}$$

Now choose $\zeta = 2\sqrt{\varepsilon}/\sqrt{T+1}$. It follows that

$$-\left(\frac{j\sqrt{T+1}}{\sqrt{\varepsilon}}\right)\zeta + \bar{\kappa}\kappa_\varepsilon\zeta^2 \leqslant 4\bar{\kappa} - 2j.$$

Therefore,

$$\sum_{j=1}^{\infty} e^j e^{4\bar{\kappa}-2j} = e^{4\bar{\kappa}} \sum_{j=1}^{\infty} e^{-j} = \frac{e^{4\bar{\kappa}}}{e-1} < \infty,$$

which implies (3.18). Hence, (3.15) follows. The desired exponential bound is obtained. □

Using the notation in the proof of Theorem 3.1, we have the following corollary.

Corollary 3.2. *If* $|\rho_i| \leqslant K_T O_1(\varepsilon + \lambda^k)$, *then for*

$$0 \leqslant c_T \leqslant \frac{1-\lambda}{K_T(|\beta|_T + 1)},$$

and $1 \leqslant \kappa_\varepsilon \leqslant T/\varepsilon$, *we have*

$$\exp\left(\frac{c_T\sqrt{\varepsilon}}{(T+1)^{3/2}}\left|\sum_{k=0}^{\kappa_\varepsilon-1}\left(\sum_{j=0}^{k}\Delta w_j^\varepsilon \rho_{k-j}\right)\beta_{k+1}\right|\right) \leqslant K.$$

4. Exponential bounds for recurrent chains

This section is devoted to the exponential bounds for singularly perturbed Markov chains, in which the fast changing part of the transition probability matrix includes l recurrent classes. That is, all states are recurrent. In this case, the state space is decomposed to

$$\begin{aligned}\mathcal{M} &= \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_l \\ &= \{s_{11}, \ldots, s_{1m_1}\} \cup \{s_{21}, \ldots, s_{2m_2}\} \cup \cdots \cup \{s_{l1}, \ldots, s_{lm_l}\}.\end{aligned} \tag{4.1}$$

Let $\{\beta_k\}$ be a sequence of matrices with real entries such that

$$\beta_k = \operatorname{diag}\big(\beta_{k,s_{11}}, \ldots, \beta_{k,s_{1m_1}}, \ldots, \beta_{k,s_{l1}}, \ldots, \beta_{k,s_{lm_l}}\big).$$

Redefine

$$\begin{aligned} n_k^\varepsilon = \sqrt{\varepsilon}\sum_{k_1=1}^{k} \big(& I_{\{\alpha_{k_1}^\varepsilon = s_{11}\}} - \nu_1^1 I_{\{\bar{\alpha}_{k_1}^\varepsilon = 1\}}, \ldots, I_{\{\alpha_{k_1}^\varepsilon = s_{1m_1}\}} - \nu_{m_1}^1 I_{\{\bar{\alpha}_{k_1}^\varepsilon = 1\}}, \\ & \ldots, I_{\{\alpha_{k_1}^\varepsilon = s_{l1}\}} - \nu_1^l I_{\{\bar{\alpha}_{k_1}^\varepsilon = l\}}, \ldots, I_{\{\alpha_{k_1}^\varepsilon = s_{lm_l}\}} - \nu_{m_l}^l I_{\{\bar{\alpha}_{k_1}^\varepsilon = l\}}\big)\beta_{k_1}. \end{aligned} \tag{4.2}$$

Let c_T be a constant such that

$$0 \leqslant c_T \leqslant \frac{1-\lambda}{K_T(|\beta|_T + 1)}$$

where

$$K_T = \sup_{\varepsilon,k} \frac{(P^\varepsilon)^k - \Phi(\varepsilon k)}{\varepsilon + \lambda^k}$$

and $\Phi(\varepsilon k)$ is given in (2.7). The result on exponential bounds is stated as follows.

Theorem 4.1. *Assume condition* (A). *Then there exists an* $\varepsilon_0 > 0$ *such that for all* $0 \leqslant \varepsilon \leqslant \varepsilon_0$, *the following error bound holds*:

$$\sup_{1\leqslant \kappa_\varepsilon \leqslant T/\varepsilon} E \exp\left(\frac{c_T}{(T+1)^{3/2}}\left|n^\varepsilon_{\kappa_\varepsilon}\right|\right) \leqslant K, \tag{4.3}$$

where n^ε_k *is defined in* (4.2), K *is a constant independent of* ε *and* T, *and* c_T *is a constant satisfying* (3.7).

Proof. Define

$$\begin{aligned}
\chi^\varepsilon_k &= \left(I_{\{\alpha^\varepsilon_k = s_{11}\}}, \ldots, I_{\{\alpha^\varepsilon_k = s_{1m_1}\}}, \ldots, I_{\{\alpha^\varepsilon_k = s_{l1}\}}, \ldots, I_{\{\alpha^\varepsilon_k = s_{lm_l}\}}\right),\\
\tilde\chi^\varepsilon_k &= \chi^\varepsilon_k \tilde{\mathbb{1}}, \qquad \tilde{\mathbb{1}} = \operatorname{diag}(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l}),\\
\overline\chi^\varepsilon_k &= \tilde\chi^\varepsilon_k \nu, \qquad \nu = \operatorname{diag}\left(\nu^1, \ldots, \nu^l\right),
\end{aligned} \tag{4.4}$$

and define Δw^ε_k and w^ε_k as in (3.2) and (3.3). Using these notations, we have

$$\chi^\varepsilon_{k+1} = \chi^\varepsilon_0 (P + \varepsilon Q)^{k+1} + \sum_{j=0}^{k} \Delta w^\varepsilon_j (P + \varepsilon Q)^{k-j}. \tag{4.5}$$

Let

$$\eta^\varepsilon_{k+1} = \chi^\varepsilon_0\left[(P+\varepsilon Q)^{k+1} - \Phi\left(\varepsilon(k+1)\right)\right] + \sum_{j=0}^{k} \Delta w^\varepsilon_j \left[(P+\varepsilon Q)^{k-j} - \Phi\left(\varepsilon(k-j)\right)\right].$$

Then,

$$\chi^\varepsilon_{k+1} = \chi^\varepsilon_0 \Phi\left(\varepsilon(k+1)\right) + \sum_{j=0}^{k} \Delta w^\varepsilon_j \Phi\left(\varepsilon(k-j)\right) + \eta^\varepsilon_{k+1}. \tag{4.6}$$

On the other hand, multiplying from the right on both sides of (4.6) by $\tilde{\mathbb{1}}$, using $\tilde\chi^\varepsilon_k = \chi^\varepsilon_k \tilde{\mathbb{1}}$, $\Phi(\varepsilon k)\tilde{\mathbb{1}} = \tilde{\mathbb{1}}\Theta(\varepsilon k)$ (see Lemma 2.1(b)), and noting that $\nu\tilde{\mathbb{1}}$ equals to the $l \times l$ identity matrix, we have

$$\begin{aligned}
\tilde\chi^\varepsilon_{k+1} &= \chi^\varepsilon_0 \Phi\left(\varepsilon(k+1)\right)\tilde{\mathbb{1}} + \sum_{j=0}^{k} \Delta w^\varepsilon_j \Phi\left(\varepsilon(k-j)\right)\tilde{\mathbb{1}} + \eta^\varepsilon_{k+1}\tilde{\mathbb{1}}\\
&= \chi^\varepsilon_0 \tilde{\mathbb{1}}\Theta\left(\varepsilon(k+1)\right) + \sum_{j=0}^{k} \Delta w^\varepsilon_j \tilde{\mathbb{1}}\Theta\left(\varepsilon(k-j)\right) + \eta^\varepsilon_{k+1}\tilde{\mathbb{1}}.
\end{aligned}$$

Thus, we arrive at

$$\overline{\chi}_{k+1}^{\varepsilon} = \chi_0^{\varepsilon}\Phi\big(\varepsilon(k+1)\big) + \sum_{j=0}^{k} \Delta w_j^{\varepsilon}\Phi\big(\varepsilon(k-j)\big) + \eta_{k+1}^{\varepsilon}\tilde{\mathbb{1}}\nu. \tag{4.7}$$

Combining (4.6) and (4.7), we obtain

$$\chi_{k+1}^{\varepsilon} - \overline{\chi}_{k+1}^{\varepsilon} = \eta_{k+1}^{\varepsilon}(I - \tilde{\mathbb{1}}\nu). \tag{4.8}$$

In view of (b) in Proposition 2.1, we have

$$\big|(P+\varepsilon Q)^k - \Phi(\varepsilon k)\big| \leqslant K_T O_1\big(\varepsilon + \lambda^k\big), \quad \text{for } 0 \leqslant k \leqslant T/\varepsilon.$$

Let

$$\rho_k = \big((P+\varepsilon Q)^k - \Phi(\varepsilon k)\big)(I - \tilde{\mathbb{1}}\nu).$$

Then

$$|\rho_k| \leqslant K_T O_1\big(\varepsilon + \lambda^k\big).$$

We have

$$\left|\sum_{k=1}^{\kappa_\varepsilon}\big[\chi_k^{\varepsilon} - \overline{\chi}_k^{\varepsilon}\big]\beta_k\right| \leqslant \left|\sum_{k=0}^{\kappa_\varepsilon - 1}\chi_0^{\varepsilon}\rho_{k+1}\beta_{k+1}\right| + \left|\sum_{k=0}^{\kappa_\varepsilon - 1}\left(\sum_{j=0}^{k}\Delta w_j^{\varepsilon}\rho_{k-j}\right)\beta_{k+1}\right|.$$

Following Corollary 3.2 and (3.14), we obtain

$$\exp\left(\frac{\sqrt{\varepsilon}c_T}{(T+1)^{3/2}}\left|\sum_{k=1}^{\kappa_\varepsilon}\big[\chi_k^{\varepsilon} - \overline{\chi}_k^{\varepsilon}\big]\beta_k\right|\right) \leqslant K.$$

The desired result then follows. □

Remark 4.2. Define $\tilde{n}_k^{\varepsilon} = n_k^{\varepsilon} + \sqrt{\varepsilon}(\chi_0^{\varepsilon} - \nu)$. Note that the initial data satisfies $\sqrt{\varepsilon}(\chi_0^{\varepsilon} - \nu) = O(\sqrt{\varepsilon})$. We have

$$\sup_{0\leqslant\kappa_\varepsilon\leqslant T/\varepsilon} E\exp\left(\frac{c_T}{(T+1)^{3/2}}\big|\tilde{n}_{\kappa_\varepsilon}^{\varepsilon}\big|\right) \leqslant K. \tag{4.9}$$

5. Applications and examples

This section is devoted to several applications of the exponential bound results including tightness, moment bounds, and asymptotic normality.

5.1. Tightness and moment bounds

In obtaining a limit theorem for asymptotic distribution, an important step involves proving the scaled sequence of occupation measures being bounded in probability. With the exponential bounds obtained, such a probability bound can be readily obtained.

Proposition 5.1. *Suppose that the conditions of Theorem* 4.1 *are fulfilled. Then for any* $\delta > 0$, *there exists a* $K_\delta > 0$ *such that for all* $0 \leqslant k \leqslant T/\varepsilon$,

$$\sup_{\varepsilon} P\big(\big|n_k^\varepsilon\big| > K_\delta\big) < \delta. \tag{5.1}$$

Proof. By virtue of the Markov inequality and Theorem 4.1,

$$\begin{aligned} P\big(\big|n_k^\varepsilon\big| > K_\delta\big) &\leqslant P\bigg(\exp\bigg(\frac{c_T}{(T+1)^{3/2}}\big|n_k^\varepsilon\big|\bigg) > \exp\bigg(\frac{c_T K_\delta}{(T+1)^{3/2}}\bigg)\bigg) \\ &\leqslant \exp\bigg(-\frac{c_T K_\delta}{(T+1)^{3/2}}\bigg) E\exp\bigg(\frac{c_T}{(T+1)^{3/2}}\big|n_k^\varepsilon\big|\bigg) \\ &\leqslant K\exp\bigg(-\frac{c_T K_\delta}{(T+1)^{3/2}}\bigg). \end{aligned}$$

It then follows that

$$\sup_{\varepsilon} P\big(\big|n_k^\varepsilon\big| > K_\delta\big) \leqslant K\exp\bigg(-\frac{c_T K_\delta}{(T+1)^{3/2}}\bigg). \tag{5.2}$$

To make the right side of (5.2) be less than δ, it suffices to have $K_\delta > (T+1)^{3/2} \times \log(K/\delta)/c_T$, which yields (5.1). □

Proposition 5.2. *Under the conditions of Theorem* 4.1, *for any* $0 < \ell < \infty$,

$$E\big|n_k^\varepsilon\big|^\ell \leqslant \frac{K(T+1)^{(3\ell)/2}\ell!}{(c_T)^\ell}, \tag{5.3}$$

where K *is given in Theorem* 4.1.

Proof. To verify this assertion, for any $z \in \mathbb{R}$, we have $\exp|z| \geqslant |z|^\ell/\ell!$. Thus

$$\big|n_k^\varepsilon\big|^\ell \leqslant \ell!\exp\bigg(\frac{c_T}{(T+1)^{3/2}}|n_k^\varepsilon|\bigg)\frac{(T+1)^{(3\ell)/2}}{c_T^\ell}.$$

Taking expectation and using Theorem 4.1, we obtain (5.3). □

5.2. Additional examples

Example 5.3. As an application of the exponential bounds, we can derive a central limit result. For simplicity, consider a fixed $i \in \mathcal{M}$, and set $N^\varepsilon = (\sqrt{\varepsilon}/\sqrt{T})\sum_{j=0}^{T/\varepsilon-1}\xi_j$, where $\xi_j = I_{\{\alpha_j^\varepsilon = i\}} - \nu_i$. Here, we have suppressed the i-dependence in both N^ε and ξ_j. Suppose that the Markov chain α_k^ε has transition matrix $P^\varepsilon = P + \varepsilon Q$ with P being an irreducible matrix. Then N^ε converges in distribution to a normal random vector with mean 0 and variance σ^2, where $\sigma^2 = \nu_i\psi_{ii}(0) + 2\nu_i\sum_{j=1}^\infty \psi_{ii}(j)$, with the $\psi_{ij}(k)$ being the initial layer correction term as given in Proposition 2.1. In view of Theorem 3.1, the characteristic function $G^\varepsilon(z) = E\exp(\imath N^\varepsilon z)$ exists for all $z \in \mathbb{R}$, where $\imath$ is the imaginary number satisfying $\imath^2 = -1$. Since P is irreducible, α_k^ε is a ϕ-mixing process with exponential mixing rate. By use of mixing inequality, we can then verify thus, $G^\varepsilon(z) \to \exp(-\sigma^2 z^2/2)$ as $\varepsilon \to 0$, and hence conclude that N^ε has a normal limit distribution. Note that the above example is for simple illustration only. In our recent work, by defining $\tilde{n}^\varepsilon(t) = \tilde{n}_k^\varepsilon$ for $t \in [\varepsilon k, \varepsilon k + \varepsilon)$, we have shown that $\tilde{n}^\varepsilon(\cdot)$ converges weakly to a diffusion process or a switching diffusion process depending on if P given in (2.1) is irreducible or having the form (2.2).

Example 5.4. In production planning with long-run average cost problems, a finite-state Markov chain is often used to characterize the underlying machine capacity and demand rate processes. Let $\{\alpha_k^\varepsilon\}$ denote a function of the machine state process representing the capacity of the machine, and let τ denote the first time when the sum $\sum_{k=0}^n (\alpha_n^\varepsilon - d) = \mu_0$ for given d and μ_0, where d is a constant demand rate and μ_0 is a measure of the accumulative difference of the demand and the capacity. It is useful to provide estimate on first and second moments of τ in terms of μ_0. In this connection, the exponential bound obtained in Theorem 4.1 is crucial in deriving such finite moments; see [16] for details.

6. Extensions and remarks

6.1. Extensions

So far, we have focused on time-homogeneous Markov chains. In fact, the exponential bounds obtained can be extended to certain nonhomogeneous problems. We shall assume that P_k^ε is time varying and takes the form

$$P_k^\varepsilon = P(\varepsilon k) + \varepsilon Q(\varepsilon k), \tag{6.1}$$

where $P(\cdot)$ and $Q(\cdot)$ are time-varying transition matrix and generator, respectively. A time-varying $m \times m$ transition matrix $P(\varepsilon k)$ is said to be weakly irreducible, if the system of equations

$$y(P(\varepsilon k) - I) = 0, \qquad y\mathbb{1}_m = 1 \tag{6.2}$$

has a unique nonnegative solution. In addition, if $P(\varepsilon k)$ is aperiodic for each k, the unique solution is termed a quasi-stationary distribution.

In addition, we also consider $P(\varepsilon k)$ having the following form:

$$P(\varepsilon k) = \begin{pmatrix} P_1(\varepsilon k) & & & & \\ & P_2(\varepsilon k) & & & \\ & & \ddots & & \\ & & & P_l(\varepsilon k) & \\ P_{*,1}(\varepsilon k) & P_{*,2}(\varepsilon k) & \cdots & P_{*,l}(\varepsilon k) & P_*(\varepsilon k) \end{pmatrix}, \tag{6.3}$$

where for each $i \leqslant l$, $P_i(\varepsilon k)$ is a transition matrix within the ith weakly irreducible class, and the last row $(P_{*,1}(\varepsilon k), \dots, P_{*,l}(\varepsilon k), P_*(\varepsilon k))$ in (6.3) corresponds to the transient states. A Markov chain with transition matrix given by (2.2) consists of l recurrent classes, whereas a Markov chain with transition matrix (6.3) has l weakly irreducible classes and a number of transient states. Let $\mathcal{M}_i = \{s_{i1}, \dots, s_{im_i}\}$ denote the states corresponding to the ith block P_i and let $\mathcal{M}_* = \{s_{*1}, \dots, s_{*m_*}\}$ denote the transient states. Then corresponding to (6.3), the state space of α_k^ε is $\mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l \cup \mathcal{M}_*$. We treat the case that in addition to the l ergodic classes, there is a group of transient states. This together with the result of the last section takes care of most of the practical concerns of a finite-state Markov chain since in a finite-state Markov chain, either all states are recurrent or it contains transient states in addition to the recurrent states.

Consider the singularly perturbed Markov chain with transition matrix (6.1) such that P_k^ε is specified by (6.3) with time-varying entries. To proceed, we modify condition (A) slightly.

(A′) For each fixed k, $Q(\varepsilon k)$ is a generator, and $P^\varepsilon(\varepsilon k)$, $P(\varepsilon k)$, and $P_i(\varepsilon k)$ for $i = 1, \ldots, l$ are transition probability matrices such that each $P_i(\varepsilon k)$ is aperiodic and weakly irreducible. On $[0, T]$, the matrix-valued function $P(t)$ is twice continuously differentiable and $Q(t)$ is Lipschitz continuous.
In addition, for each fixed k, with t denoting εk, there exists an $m_* \times m_*$ nonsingular matrix $B(t)$ and constant matrices P_* and $P_{*,i}$ satisfying $P_*(t) - I = B(t)(P_* - I)$, and $P_{*,i}(t) = B(t)P_{*,i}$, for $i = 1, \ldots, l$. All the eigenvalues of $P_*(t)$ are inside the unit circle.

It follows from condition (A′), the following quantities

$$a_i(\varepsilon k) = -\big(P_*(\varepsilon k) - I\big)^{-1} P_{*,i}(\varepsilon k)\mathbb{1}_{m_i} = -(P_* - I)P_{*,i}\mathbb{1}_{m_i} = a_i,\quad \text{for } i = 1, \ldots, l,$$

are independent of time. Thus

$$A_*(\varepsilon k) = \big(a_1(\varepsilon k), \ldots, a_l(\varepsilon k)\big) = A_* = (a_1, \ldots, a_l) \in \mathbb{R}^{m_* \times l}. \tag{6.4}$$

Partition the matrix $Q(\varepsilon k)$ as

$$Q(\varepsilon k) = \begin{pmatrix} Q^{11}(\varepsilon k) & Q^{12}(\varepsilon k) \\ Q^{21}(\varepsilon k) & Q^{22}(\varepsilon k) \end{pmatrix}, \tag{6.5}$$

where

$$Q^{11}(\varepsilon k) \in \mathbb{R}^{(m-m_*)\times(m-m_*)}, \qquad Q^{12}(\varepsilon k) \in \mathbb{R}^{(m-m_*)\times m_*},$$
$$Q^{21}(\varepsilon k) \in \mathbb{R}^{m_*\times(m-m_*)}, \quad \text{and} \quad Q^{22}(\varepsilon k) \in \mathbb{R}^{m_*\times m_*}.$$

Write

$$\begin{aligned} \overline{Q}(\varepsilon k) &= \operatorname{diag}\big(\nu^1(\varepsilon k), \ldots, \nu^l(\varepsilon k)\big)\big(Q^{11}(\varepsilon k)\tilde{\mathbb{1}} + Q^{12}(\varepsilon k)A_*\big), \\ \overline{Q}_*(\varepsilon k) &= \nu_*(\varepsilon k)Q(\varepsilon k)\tilde{\mathbb{1}}_*. \end{aligned} \tag{6.6}$$

Define

$$\tilde{\mathbb{1}}_* = \begin{pmatrix} \tilde{\mathbb{1}} & 0_{(m-m_*)\times m_*} \\ A_* & 0_{m_*\times m_*} \end{pmatrix}, \qquad \nu_*(\varepsilon k) = \operatorname{diag}\big(\nu^1(\varepsilon k), \ldots, \nu^l(\varepsilon k), 0_{m_*\times m_*}\big), \tag{6.7}$$

and define sequences of centered occupation measures by

$$o_{k,ij}^\varepsilon = \begin{cases} \varepsilon \sum_{r=0}^{k-1}(I_{\{\alpha_r^\varepsilon = s_{ij}\}} - \nu_j^i(\varepsilon r)I_{\{\overline{\alpha}_r^\varepsilon = i\}}), & \text{for } i = 1, \ldots, l, \\ \varepsilon \sum_{r=0}^{k-1} I_{\{\alpha_r^\varepsilon = s_{*j}\}}, & \text{for } i = *. \end{cases} \tag{6.8}$$

The following results can be derived. The proof of part (a) is in [21], whereas the proof of part (b) can be found in [23].

Lemma 6.1. *Under* (A′), *the following assertions hold*:

(a) $x_k^\varepsilon = (\theta(\varepsilon k)\,\mathrm{diag}(\nu^1(\varepsilon k), \ldots, \nu^l(\varepsilon k)), 0_{m_*}) + O(\varepsilon + \lambda^k)$, *where* $0_{m_*} \in \mathbb{R}^{1\times m_*}$ *and* $\theta(t) = (\theta_1(t), \ldots, \theta_l(t)) \in \mathbb{R}^{1\times l}$ *satisfies*

$$\frac{d\theta(t)}{dt} = \theta(t)\overline{Q}(t), \qquad \theta_i(0) = x^i(0)\mathbb{1}_{m_i} - x^*(0)a_i.$$

The transition matrix satisfies

$$P^\varepsilon(k_0, k) = \Phi(\varepsilon k_0, \varepsilon k) + \Psi(k_0, k) + \varepsilon\hat{\Phi}(\varepsilon k_0, \varepsilon k) + \varepsilon\hat{\Psi}(k_0, k) + O\big(\varepsilon^2\big), \quad \text{for } k \leqslant T/\varepsilon,$$

for some λ *with* $0 < \lambda < 1$, *where*

$$\Phi(t_0, t) = \tilde{\mathbb{1}}_*\Theta_*(t_0, t)\nu_*(t) \quad \text{with } \Theta_*(t_0, t) = \mathrm{diag}\big(\Theta(t_0, t), I_{m_*\times m_*}\big), \tag{6.9}$$

where $\Theta(t_0, t) = (\theta_{ij}(t_0, t))$ *satisfies the differential equation*

$$\frac{\partial\Theta(t_0, t)}{\partial t} = \Theta(t_0, t)\overline{Q}(t), \qquad \Theta(t_0, t_0) = I.$$

(b) *For each* $j = 1, \ldots, m_i$,

$$\sup_{0\leqslant k\in T/\varepsilon} E\big|o_{k,ij}^\varepsilon\big|^2 = \begin{cases} O(\varepsilon), & \text{for } i = 1, \ldots, l, \\ O(\varepsilon^2), & \text{for } i = *. \end{cases}$$

In Lemma 6.1, the transition probability matrix $P^\varepsilon(j, k)$ for $0 \leqslant j \leqslant k$ is defined as $P^\varepsilon(j, k) = P(\varepsilon j)P(\varepsilon(j + 1)) \cdots P(\varepsilon k)$, $P^\varepsilon(k + 1, k) = I$. This representation is more complex than that of the constant case in Sections 3 and 4, since it now also depends on the initial time. In addition, since the matrices generally do not commute, the order of the product is important. To obtain the desired exponential upper bounds, using (6.7), define

$$\begin{aligned}
\chi_k^\varepsilon &= \big(I_{\{\alpha_k^\varepsilon = s_{11}\}}, \ldots, I_{\{\alpha_k^\varepsilon = s_{1m_1}\}}, \ldots, I_{\{\alpha_k^\varepsilon = s_{l1}\}}, \ldots, I_{\{\alpha_k^\varepsilon = s_{lm_l}\}}, \\
&\qquad I_{\{\alpha_k^\varepsilon = s_{*1}\}}, \ldots, I_{\{\alpha_k^\varepsilon = s_{*m_*}\}}\big), \\
\tilde{\chi}_k^\varepsilon &= \chi_k^\varepsilon\tilde{\mathbb{1}}_*, \qquad \overline{\chi}_k^\varepsilon = \tilde{\chi}_k^\varepsilon\nu_*(t),
\end{aligned} \tag{6.10}$$

and define Δw_k^ε and w_k^ε as in the previous section. Using the notation of a partitioned vector $v = (v^1, v^2, \ldots, v^l, v^*)$ with $v^i \in \mathbb{R}^{1\times m_i}$, it is readily seen that

$$\begin{aligned}
\chi_k^\varepsilon &= \big(I_k^\varepsilon\mathbb{1}_{m_1} + I_k^{\varepsilon,*}a_1, \ldots, I_k^{\varepsilon,l}\mathbb{1}_{m_l} + I_k^{\varepsilon,*}a_l, 0_{m_*}\big), \\
\overline{\chi}_k^\varepsilon &= \big(\big(I_k^\varepsilon\mathbb{1}_{m_1} + I_k^{\varepsilon,*}a_1\big)\nu^1(\varepsilon k), \ldots, \big(I_k^{\varepsilon,l}\mathbb{1}_{m_l} + I_k^{\varepsilon,*}a_l\big)\nu^l(\varepsilon k), 0_{m_*}\big),
\end{aligned}$$

where 0_{m_*} is an $\mathbb{R}^{1\times m_*}$ zero vector. As in (4.8), we can derive

$$\chi_{k+1}^\varepsilon - \overline{\chi}_{k+1}^\varepsilon = \eta_{k+1}^\varepsilon\big(I - \tilde{\mathbb{1}}_*\nu_*(\varepsilon k)\big), \tag{6.11}$$

where

$$\eta_{k+1}^{\varepsilon} = \chi_0^{\varepsilon}\big[P^{\varepsilon}(0, k+1) - \Phi\big(0, \varepsilon(k+1)\big)\big] + \sum_{j=0}^{k} \Delta w_j^{\varepsilon}\big[P^{\varepsilon}(j, k) - \Phi(\varepsilon j, \varepsilon k)\big].$$

The rest of the development follows from the same line of argument as that of Theorem 4.1. We thus arrive at the following result.

Theorem 6.2. *Assume condition* (A$'$). *Then the conclusion of Theorem* 4.1 *continue to hold with*

$$K_T = \sup_{\varepsilon, k_0, k} \frac{P^{\varepsilon}(k_0, k) - \Phi(\varepsilon k_0, \varepsilon k)}{\varepsilon + \lambda^{k-k_0}}.$$

6.2. Further remarks

This paper has been devoted to obtaining exponential type bounds for a sequence of scaled and centered occupation measures. The results obtained can be used to help us in the development of infinite horizon stochastic control problems and to further our understanding of hybrid systems involving singularly perturbed Markov chains. Future work can be directed to the large deviations study.

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